NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE SECOND DERIVATIVES ABSOLUTE VALUES ARE CONVEX AND QUASI-CONVEX

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Abstract. In this paper, we establish several new inequalities for twice differantiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality. Some applications for special means of real numbers are also provided.

1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [9]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if whenever $x, y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

This definition has its origins in Jensen's results from [5] and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, canvex analysis. Convex curvers and convex bodies have appeared in mathematical literature since antiquity and there are many important resuls related to them. We say that f is concave if $(-f)$ is convex.

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f(tx + (1 - t)y) \leq \sup\{f(x), f(y)\}$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [4]).

For several recent results concerning Hermite-Hadamard integral inequality, we refer the reader to ([1]-[8]).

In [7] some inequalities of Hermite-Hadamard type for differentiable convex mappings were proved using the following lemma.

2000 *Mathematics Subject Classification.* 26D15,
Key words and phrases. Hermite-Hadamard inequality, convex function, quasi-convex function.
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Lemma 1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$. If $f' \in L([a, b])$, then we have*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \\ &= (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)\, f'(ta+(1-t)b)dt\right]. \end{aligned} \tag{1.2}$$

One more general result related to (1.2) was established in [8]. The main result in [7] is as follows:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I$ with $a < b$. If the mapping $|f'|$ is convex on $[a, b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left(\frac{|f'(a)| + |f'(b)|}{2}\right). \tag{1.3}$$

In [6], Pearce and Pečarić proved the following theorem.

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If the mapping $|f'|^q$ is convex on $[a, b]$ for some $q \ge 1$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{1.4}$$

In this article, using functions whose second derivatives absolute values are convex and quasi-convex, we obtained new inequalities releted to the left side of Hermite-Hadamard inequality. Finally, we gave some applications for special means of real numbers.

2. Hermite-Hadamard type inequalities for convex functions

We will establish some new results connected with the left-hand side of (1.1) used the following Lemma. Now, we give the following new Lemma for our results:

Lemma 2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1[a, b]$, then*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2}) \\ &\qquad = \frac{(b-a)^2}{2}\int_0^1 m(t)\left[f''(ta+(1-t)b) + f''(tb+(1-t)a)\right]dt, \end{aligned}$$

where

$$m(t) := \begin{cases} t^2 & , t \in [0, \frac{1}{2}) \\ (1-t)^2 & , t \in [\frac{1}{2}, 1]. \end{cases}$$

Proof. It suffices to note that

$$\begin{aligned}
I_1 &= \int_0^1 m(t)\, f''(ta+(1-t)b)dt \\
&= \int_0^{1/2} t^2 f''(ta+(1-t)b)dt + \int_{1/2}^1 (1-t)^2\, f''(ta+(1-t)b)dt \\
&= \frac{1}{a-b} t^2 f'(ta+(1-t)b) \Big|_0^{1/2} - \frac{2}{a-b}\int_0^{1/2} tf'(ta+(1-t)b)dt \\
&+ \frac{1}{a-b}(1-t)^2 f'(ta+(1-t)b) \Big|_{1/2}^{1} + \frac{2}{a-b}\int_{1/2}^1 (1-t)\, f'(ta+(1-t)b)dt \\
&= -\frac{1}{4(b-a)} f'(\frac{a+b}{2}) + \frac{2}{b-a}\left[\frac{1}{a-b} tf(ta+(1-t)b) \Big|_0^{1/2} - \frac{1}{a-b}\int_0^{1/2} f(ta+(1-t)b)dt\right] \\
&+ \frac{1}{4(b-a)} f'(\frac{a+b}{2}) - \frac{2}{b-a}\left[\frac{1}{a-b}(1-t)\, f(ta+(1-t)b) \Big|_{1/2}^{1} + \frac{1}{a-b}\int_{1/2}^1 f(ta+(1-t)b)dt\right] \\
&= \frac{2}{b-a}\left[-\frac{1}{2(b-a)} f(\frac{a+b}{2}) + \frac{1}{b-a}\int_0^{1/2} f(ta+(1-t)b)dt\right] \\
&- \frac{2}{b-a}\left[\frac{1}{2(b-a)} f(\frac{a+b}{2}) - \frac{1}{b-a}\int_{1/2}^1 f(ta+(1-t)b)dt\right] \\
&= -\frac{2}{(b-a)^2} f(\frac{a+b}{2}) + \frac{2}{(b-a)^2}\int_0^1 f(ta+(1-t)b)dt.
\end{aligned}$$

Using the change of the variable $x = ta + (1-t)b$ for $t \in [0,1]$, which gives

$$I_1 = -\frac{2}{(b-a)^2} f(\frac{a+b}{2}) + \frac{2}{(b-a)^3}\int_a^b f(x)dx. \tag{2.1}$$

Similarly, we can show that

$$\begin{aligned}
I_2 &= \int_{1/2}^1 m(t)\, f''(tb+(1-t)a)dt \\
&= \int_0^{1/2} t^2 f''(tb+(1-t)a)dt + \int_{1/2}^1 (1-t)^2\, f''(tb+(1-t)a)dt \qquad (2.2) \\
&= -\frac{2}{(b-a)^2} f(\frac{a+b}{2}) + \frac{2}{(b-a)^3}\int_a^b f(x)dx.
\end{aligned}$$

Thus, summing the equalities (2.1) and (2.2), and multiplying the both sides by $\frac{(b-a)^2}{4}$, we obtain

$$\frac{(b-a)^2}{4}(I_1+I_2) = \frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})$$

which is required. $\square$

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° with $f'' \in L_1[a,b]$. If $|f''|$ is convex on $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \le \frac{(b-a)^2}{24}\left[\frac{|f''(a)|+|f''(b)|}{2}\right]. \tag{2.3}$$

Proof. From Lemma 2 and the convexity of $|f''|$, it follows that

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \\ &\le \frac{(b-a)^2}{4}\left\{\int_0^1 |m(t)|\,|f''(ta+(1-t)b)|\,dt + \int_0^1 |m(t)|\,|f''(tb+(1-t)a)|\,dt\right\} \\ &\le \frac{(b-a)^2}{4}\left\{\int_0^1 |m(t)|\left[t\,|f''(a)|+(1-t)\,|f''(b)|\right]dt + \int_0^1 |m(t)|\left[t\,|f''(b)|+(1-t)\,|f''(b)|\right]dt\right\}. \end{aligned} \tag{2.4}$$

By simple computation,

$$\begin{aligned} &\int_0^1 m(t)\left[t\,|f''(a)|+(1-t)\,|f''(b)|\right]dt \\ &\le \int_0^{1/2} t^2\left[t\,|f''(a)|+(1-t)\,|f''(b)|\right]dt + \int_{1/2}^1 (1-t)^2\left[t\,|f''(a)|+(1-t)\,|f''(b)|\right]dt \\ &= \frac{|f''(a)|+|f''(b)|}{24} \end{aligned} \tag{2.5}$$

and similarly,

$$\int_0^1 m(t)\left[t\,|f''(b)|+(1-t)\,|f''(a)|\right]dt = \frac{|f''(a)|+|f''(b)|}{24}. \tag{2.6}$$

Using (2.5) and (2.6) in (2.4), we obtain (2.3). $\square$

Remark 1. *We note that the obtained midpoint inequality (2.3) is better than the inequality (1.3).*

Another similar result may be extended in the following theorem.

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a,b \in I$, $a<b$. If $|f''|^q$ is convex on $[a,b]$, $q>1$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \le \frac{(b-a)^2}{8\,(2p+1)^{1/p}}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{1/q}. \tag{2.7}$$

Proof. From Lemma 2 and using well known Hölder's integral inequality , we get,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right|$$

$$\leq \frac{(b-a)^2}{4}\left(\int_0^1 |m(t)|^p\, dt\right)^{1/p}\left\{\left(\int_0^1 |f''(ta+(1-t)b)|^q\, dt\right)^{1/q}\right.$$

$$\left.+\left(\int_0^1 |f''(tb+(1-t)a)|^q\, dt\right)^{1/q}\right\}.$$

Since $|f''|^q$ is convex on $[a,b]$, we known that for $t\in[0,1]$

$$|f''(ta+(1-t)b)|^q \leq t\,|f''(a)|^q + (1-t)\,|f''(b)|^q\,.$$

Hence,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right|$$

$$\leq \frac{(b-a)^2}{16\,(2p+1)^{1/p}}\left\{\left(\int_0^1 \left[t\,|f''(a)|^q+(1-t)\,|f''(b)|^q\right] dt\right)^{\frac{1}{q}}+\left(\int_0^1 \left[t\,|f''(b)|^q+(1-t)\,|f''(a)|^q\right] dt\right)^{\frac{1}{q}}\right\}$$

$$=\frac{(b-a)^2}{8\,(2p+1)^{1/p}}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{1/q},$$

where we have used the fact that

$$\int_0^1 |m(t)|^p\, dt = \int_0^{1/2} t^{2p}dt + \int_{1/2}^1 (1-t)^{2p}\, dt = \frac{1}{4^p\,(2p+1)}$$

which completes the proof. □

An improvement of the constants in Theorem 4 and a consolidation of this result with Theorem 3 are given in the following theorem.

Theorem 5. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on I° such that $f''\in L_1[a,b]$ where $a,b\in I$, $a<b$. If $|f''|^q$ is convex on $[a,b]$, $q\geq 1$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \leq \frac{(b-a)^2}{24}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{2.8}$$

Proof. From Lemma 2 and using well known power mean inequality , we get,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right|$$

$$\leq \frac{(b-a)^2}{4}\left(\int_0^1 |m(t)|\,dt\right)^{1/p}\left\{\left(\int_0^1 |m(t)|\,|f''(ta+(1-t)b)|^q\,dt\right)^{1/q}\right.$$

$$\left.+\left(\int_0^1 |m(t)|\,|f''(tb+(1-t)a)|^q\,dt\right)^{1/q}\right\}.$$

Since $|f''|^q$ is convex on $[a,b]$, we known that for $t\in[0,1]$

$$|f''(ta+(1-t)b)|^q \leq t\,|f''(a)|^q + (1-t)\,|f''(b)|^q\,.$$

Hence,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right|$$

$$\leq \frac{(b-a)^2}{4}\frac{1}{(12)^{1/p}}\left\{\left(\int_0^{1/2} t^2\left[t\,|f''(a)|^q + (1-t)\,|f''(b)|^q\right]dt\right.\right.$$

$$\left.+\int_{1/2}^1 (1-t)^2\left[t\,|f''(a)|^q + (1-t)\,|f''(b)|^q\right]dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^{1/2} t^2\left[t\,|f''(b)|^q + (1-t)\,|f''(a)|^q\right]dt + \int_{1/2}^1 (1-t)^2\left[t\,|f''(b)|^q + (1-t)\,|f''(a)|^q\right]dt\right)^{1/q}\right\}$$

$$= \frac{(b-a)^2}{4}\frac{2}{(12)^{1/p}}\left[\frac{|f''(a)|^q + |f''(b)|^q}{24}\right]^{1/q},$$

where we have used the fact that

$$\int_0^1 |m(t)|\,dt = \int_0^{1/2} t^2 dt + \int_{1/2}^1 (1-t)^2\,dt = \frac{1}{12}$$

which completes the proof. □

Remark 2. *For $q=1$, this theorem reduces Theorem 3. For $q=p/(p-1)$, $p>1$, we have an improvement of the constants in Theorem 4, since $3^p>(2p+1)$ if $p>1$ and accordingly*

$$\frac{1}{24} < \frac{1}{8(2p+1)^{\frac{1}{p}}}.$$

Remark 3. *We note that the obtained midpoint inequality (2.8) is better than the inequality (1.4).*

3. Hermite-Hadamard type inequalities for quasi-convex functions

Theorem 6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a, b \in I$, $a < b$. If $|f''|$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \le \frac{(b-a)^2}{24}\sup\left\{\left|f''(a)\right|, \left|f''(b)\right|\right\}. \tag{3.1}$$

Proof. From Lemma 2, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \\
&\le \frac{(b-a)^2}{4}\int_0^1 |m(t)| \left[|f''(ta+(1-t)b)| + |f''(tb+(1-t)a)|\right] dt \\
&\le \frac{(b-a)^2}{4} 2\left[\int_0^{1/2} t^2 \sup\left\{|f''(a)|, |f''(b)|\right\} dt + \int_{1/2}^1 (1-t)^2 \sup\left\{|f''(b)|, |f''(a)|\right\} dt\right] \\
&= \frac{(b-a)^2}{24}\sup\left\{|f''(a)|, |f''(b)|\right\}.
\end{aligned}$$

□

Therefore, we can deduce the following result for quasi-convex functions.

Corollary 1. *Let f be as in Theorem 6. Additionally, if*

1° $|f''|$ is increasing, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \le \frac{(b-a)^2}{24}\left|f''(b)\right|. \tag{3.2}$$

2° $|f''|$ is decreasing, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \le \frac{(b-a)^2}{24}\left|f''(a)\right|. \tag{3.3}$$

Proof. It follows directly by Theorem 6. □

Theorem 7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$, $a < b$. If $|f''|^q$ is quasi-convex on $[a,b]$, $q \ge 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \le \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{1/q} \tag{3.4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2, using the well known Hölder's integral inequality , we have,

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \\
&\leq \frac{(b-a)^2}{4}\left(\int_0^1 |m(t)|^p\,dt\right)^{1/p}\left\{\left(\int_0^1 |f''(ta+(1-t)b)|^q\,dt\right)^{1/q}\right. \\
&\left. + \left(\int_0^1 |f''(tb+(1-t)a)|^q\,dt\right)^{1/q}\right\}. \\
&\leq \frac{(b-a)^2}{16(2p+1)^{\frac{1}{p}}}\left\{\left(\int_0^1 \sup\left\{|f''(a)|^q, |f''(b)|^q\right\}dt\right)^{1/q} + \left(\int_0^1 \sup\left\{|f''(b)|^q, |f''(a)|^q\right\}dt\right)^{1/q}\right\} \\
&= \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{1/q}.
\end{aligned}$$

where we have used the fact that

$$\int_0^1 |m(t)|^p\,dt = \int_0^{1/2} t^{2p}dt + \int_{1/2}^1 (1-t)^{2p}\,dt = \frac{1}{4^p(2p+1)}$$

which completes the proof. □

Theorem 8. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$, $a < b$. If $|f''|^q$ is quasi-convex on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \leq \frac{(b-a)^2}{24}\left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{1/q}. \tag{3.5}$$

Proof. From Lemma 2, using the well known power mean inequality , we have,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right|$$

$$\leq \frac{(b-a)^2}{4}\left(\int_0^1 |m(t)|\, dt\right)^{1/p}\left\{\left(\int_0^1 |m(t)|\,|f''(ta+(1-t)b)|^q\, dt\right)^{1/q}\right.$$

$$\left.+\left(\int_0^1 |m(t)|\,|f''(tb+(1-t)a)|^q\, dt\right)^{1/q}\right\}.$$

$$\leq \frac{(b-a)^2}{4}\frac{1}{(12)^{1/p}}\left\{\left(\int_0^{1/2} t^2 \sup\left\{|f''(a)|^q, |f''(b)|^q\right\} dt\right.\right.$$

$$\left.+\int_{1/2}^1 (1-t)^2 \sup\left\{|f''(a)|^q, |f''(b)|^q\right\} dt\right)^{1/q}$$

$$\left.+\left(\int_0^{1/2} t^2 \sup\left\{|f''(b)|^q, |f''(a)|^q\right\} dt + \int_{1/2}^1 (1-t) \sup\left\{|f''(b)|^q, |f''(a)|^q\right\} dt\right)^{1/q}\right\}$$

$$= \frac{(b-a)^2}{4}\frac{1}{(12)^{1/p}}\frac{2}{(12)^{1/q}}\left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{1/q}.$$

□

Remark 4. *For $q=1$, this theorem reduces Theorem 6. For $q=p/(p-1)$, $p>1$, we have an improvement of the constants in Theorem 7, since $3^p > (2p+1)$ if $p>1$ and accordingly*

$$\frac{1}{24} < \frac{1}{8(2p+1)^{\frac{1}{p}}}.$$

4. Applications to Some Special Means

We now consider the applications of our Theorems to the following special means:

(a) The arithmetic mean: $A = A(a,b) := \dfrac{a+b}{2}$, $a,b \geq 0$,

(b) The geometric mean: $G = G(a,b) := \sqrt{ab}$, $a,b \geq 0$,

(c) The harmonic mean:

$$H = H(a,b) := \frac{2ab}{a+b},\ a,b \geq 0,$$

(d) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & if \quad a = b \\ \frac{b-a}{\ln b - \ln a} & if \quad a \neq b \end{cases},\quad a,b > 0,$$

(e) The Identric mean:

$$I=I\left(a,b\right):=\left\{\begin{array}{cc} a & \text{if} \quad a=b \\ & \\ \frac{1}{e}\left(\frac{b^{b}}{a^{a}}\right)^{\frac{1}{b-a}} & \text{if} \quad a\neq b \end{array}\right. \quad , \quad a,b>0,$$

(f) The $p-$logarithmic mean

$$L_{p}=L_{p}(a,b):=\left\{\begin{array}{cc} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a\neq b \\ & \\ a & \text{if} \quad a=b \end{array}\right. \quad , \quad p\in\mathbb{R}\diagdown\{-1,0\}\,;\ a,b>0.$$

It is well known that L_p is monotonic nondecreasing over $p\in\mathbb{R}$ with $L_{-1}:=L$ and $L_0:=I$. In particular, we have the following inequalities

$$H\leq G\leq L\leq I\leq A.$$

The following proposition holds:

Proposition 1. *Let $a,b\in\mathbb{R}$, $0<a<b$, $n\in\mathbb{Z}$ and $|n(n-1)|\geq 3$. Then, we have*

$$\left|L_{n}^{n}\left(a,b\right)-A^{n}\left(a,b\right)\right|\leq\left|n(n-1)\right|\frac{\left(b-a\right)^{2}}{48}A\left(a^{(n-2)},b^{(n-2)}\right).$$

Proof. The proof is immediate from Theorem 3 applied for $f(x)=x^n$, $x\in\mathbb{R}$, $n\in\mathbb{Z}$ and $|n(n-1)|\geq 3$. □

Proposition 2. *Let $a,b\in(0,\infty)$ and $a<b$. Then, for all $q>1$,we have*

$$\ln\left(\frac{I\left(a,b\right)}{A\left(a,b\right)}\right)\leq\frac{\left(b-a\right)^{2}}{8a^{2}b^{2}\left(2p+1\right)^{\frac{1}{p}}}\left[A\left(a^{2q},b^{2q}\right)\right]^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 4 applied to the mapping $f:(0,\infty)\rightarrow(-\infty,0)$, $f(x)=-\ln x$ and the details are omitted. □

Proposition 3. *Let $a,b\in\mathbb{R}$, $0<a<b$ and $n\in\mathbb{Z}$, $|n(n-1)|>2$. Then, for all $q>1$,we have*

$$\left|L_{n}^{n}\left(a,b\right)-A^{n}\left(a,b\right)\right|\leq\left|n(n-1)\right|\frac{\left(b-a\right)^{2}}{24}\left[A\left(a^{q(n-2)},b^{q(n-2)}\right)\right]^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 5 applied for $f(x)=x^n$, $x\in\mathbb{R}$, $n\in\mathbb{Z}$ and $|n(n-1)|\geq 3$. □

Proposition 4. *Let $a,b\in\mathbb{R}$, $0<a<b$. Then, for all $q>1$, we have*

$$\left|L^{-1}\left(a,b\right)-A^{-1}\left(a,b\right)\right|\leq\frac{\left(b-a\right)^{2}}{24}\frac{2^{\frac{q-1}{q}}}{a^{3}b^{3}}\left[a^{3q}+b^{3q}\right]^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 5 applied to $f(x)=\frac{1}{x}$, $x\in[a,b]$ and the details are omitted. □

Proposition 5. *Let $a,b\in\mathbb{R}$, $a<b$ and $0\notin[a,b]$, then, for all $q\geq 1$, the following inequality holds:*

$$\left|L^{-1}\left(a,b\right)-A^{-1}\left(a,b\right)\right|\leq\frac{\left(b-a\right)^{2}}{24}\left(\sup\left\{\left|\frac{2}{a^{3}}\right|^{q},\left|\frac{2}{b^{3}}\right|^{q}\right\}\right)^{\frac{1}{q}}.$$

Proof. The proof is obvious from Theorem 8 applied to the quasi-convex mapping $f(x)=\frac{1}{x}$, $x\in[a,b]$. $\square$

Proposition 6. *Let $a,b\in\mathbb{R}$, $0<a<b$ and $n\in\mathbb{Z}$, $|n(n-1)|\geq 3$, then, for all $q\geq 1$, the following inequality holds:*

$$\left|L_n^n(a,b)-A^n(a,b)\right|\leq |n(n-1)|\frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\left(\sup\left\{a^{q(n-2)},b^{q(n-2)}\right\}\right)^{\frac{1}{q}}$$

Proof. The proof is obvious from Theorem 7 applied to the quasi-convex mapping $f(x)=x^n$, $x\in[a,b]$, $n\in\mathbb{Z}$ and $|n(n-1)|\geq 3$. $\square$

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